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To cite this article: Taewook Lee & Cheolwoo Park (2016): Tests for serial correlation in mean and variance of a sequence of time series objects, Journal of Statistical Computation and Simulation, DOI: 10.1080/00949655.2016.1217336

To link to this article: http://dx.doi.org/10.1080/00949655.2016.1217336

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Tests for serial correlation in mean and variance of a sequence of time series objects

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\textbf{ABSTRACT}

In this era of Big Data, large-scale data storage provides the motivation for statisticians to analyse new types of data. The proposed work concerns testing serial correlation in a sequence of sets of time series, here referred to as time series objects. An example is serial correlation of monthly stock returns when daily stock returns are observed. One could consider a representative or summarized value of each object to measure the serial correlation, but this approach would ignore information about the variation in the observed data. We develop Kolmogorov–Smirnov-type tests with the standard bootstrap and wild bootstrap Ljung–Box test statistics for serial correlation in mean and variance of time series objects, which take the variation within a time series object into account. We study the asymptotic property of the proposed tests and present their finite sample performance using simulated and real examples.

\textbf{ARTICLE HISTORY}

Received 31 March 2016
Accepted 22 July 2016

\textbf{KEYWORDS}

Autocorrelation; Kolmogorov–Smirnov test; Ljung–Box test; time series objects; wild bootstrap

\section{Introduction}

Serial correlation is a common occurrence in time series data and testing its existence is a fundamental problem in time series analysis. It serves as an elementary tool for analysing and modelling time series data using statistical methods. In financial time series, many existing works \cite{1,2} study the serial correlation of daily, weekly and monthly stock returns or indices, and they sometimes find somewhat paradoxical results \cite{3}. There could be several possible explanations for those results, but we do not dwell on that here; rather, we explore the basic question: How do we test daily, weekly or monthly autocorrelation? More generally, how do we test the serial correlation in a sequence of time series objects? Here, a time series object refers to a set of time series (e.g. stock prices recorded every 5 minutes in a day or every business days in a month). Due to the availability of large-scale computational power and storage space, financial time series are recorded every second, and they are typically aggregated into manageable size of representative numbers. For example, when measuring monthly autocorrelation, it is a common practice to use the monthly log-return of each month, which is equivalent to the sum of all daily log-returns of each month. However, it is questionable whether the use of these monthly returns is appropriate for measuring and testing the monthly serial correlation because this approach ignores any information about variability within a month. See Han et al. \cite{4} and the references therein for more details.

To motivate our work, Figure 1 displays the daily closing values of the Dow Jones index from March to April of 2013 in (a) and from September to October of 2013 in (b). It is clear that the
trends are different from each other. In this particular example, we are interested in testing monthly serial correlation when daily data are available. As a simple approach, one could use a representative value from each month, such as last day’s index or average index, and test the correlation in that time series. Alternatively, one can utilize the within-variation by creating interval-valued data using the maximum and minimum for each month as in [4–6] and articles therein. In Figure 1, the red solid vertical line indicates the closing value of the last day for each month, the black dotted line the mean value of each month, and the blue dashed line the maximum and minimum values of each month. It is evident from the plots that using different statistics results in a different relationship between the two months.

The objective of the proposed work is to study the serial correlation in mean and variance of a sequence of a set of time series – i.e. time series objects. Examples include daily (or monthly) serial correlation when time series are observed on a minutely (or daily) basis. To our knowledge, a thorough study on this topic has not been done in the time series literature. Our main contribution is to develop Kolmogorov–Smirnov-type tests [7] with Ljung–Box test statistics,[8] which utilize the internal variation information within a time series object via bootstrap sampling. As mentioned in [9], standard bootstrap procedures could be invalid in the presence of conditional heteroscedasticity. Therefore, when constructing our test statistics for serial correlation in mean, we additionally develop a test that adopts the wild bootstrap sampling method of Liu [10] and Mammen.[11] We refer to Ahlgren and Catani,[12] Catani et al. [13] and the references therein for more details of wild bootstrap. In time series objects, we note that the length of the time series can vary from object to object. This is particularly important in testing monthly serial correlation because each month is likely to have a different number of business days. Hence, we devise the proposed tests such that they can work with time series objects of different lengths. To provide theoretical justification of the proposed bootstrap tests, we study the asymptotic properties and present their finite sample performance using various simulated and real examples.

The remainder of the paper is organized as follows. Section 2 introduces weakly stationary time series objects and discusses the ordinary Ljung–Box tests for serial correlation in mean and variance of time series objects. Section 3 develops the bootstrap-based tests and describes their asymptotic properties. Section 4 presents results of simulation study and Section 5 presents real data analysis.

2. Weakly stationary time series objects

In this section, we introduce a weakly stationary time series object with fixed length and random length. Details are provided in Section 2 of supplementary materials.
A sequence of time series objects with fixed length is a vector-valued time series, denoted by \( \{X_t\}_{t=-\infty}^{\infty} \) with \( X_t = (X_{t,1}, X_{t,2}, \ldots, X_{t,d})' \). Here, \( (X_{t,1}, X_{t,2}, \ldots, X_{t,d})' \) forms a sequence of scalar valued time series with shorter time intervals than \( \{X_t\}_{t=-\infty}^{\infty} \); the time between \( X_{t,1} \) and \( X_{t,d} \) is shorter than the time between \( X_t \) and \( X_{t+1} \). Typical examples include weekly log-return time series objects with \( d = 5 \) business days each week, and daily log-return time series objects with \( d = 70 \) 5-minute records each day.

For random length, we denote a sequence of time series objects with random length \( \{D_t\}_{t=-\infty}^{\infty} \) by \( \{X_t\}_{t=-\infty}^{\infty} \) with \( X_t = (X_{t,1}, X_{t,2}, \ldots, X_{t,D_t})' \). For example, because there are different business days each month, it is necessary to adopt random length rather than fixed length in the definition of monthly log-returns time series objects. That is, a monthly log-return time series object \( X_t = (X_{t,1}, X_{t,2}, \ldots, X_{t,D_t})' \) contains \( D_t \) business days at the \( t \)th month, and its \( i \)th component \( X_{t,i} \) of \( X_t \) is the daily log-return of the \( i \)th business day of the \( t \)th month. Throughout the paper, we use the following assumption on \( \{D_t\}_{t=-\infty}^{\infty} \):

\[
\{D_t\}_{t=-\infty}^{\infty} \text{ is IID discrete random variables on a finite sample space } S_D = \{d_1, \ldots, d_K\} \text{ with } d_1 < d_2 < \cdots < d_K \text{ and independent of } \{X_{t,1}, \ldots, X_{t,D_t}\}_{t=-\infty}^{\infty} \text{ for any given } t \in S_D, -\infty < t < \infty.
\]

We also denote the probability mass function of \( D_t \) as \( f_D(x) = P[D_t = x] \) for \( x = d_1, \ldots, d_K \). Note that if \( S_D = \{d\} \), a sequence of time series objects with random length reduces to the case with fixed length. From here on, we only consider a sequence of time series objects \( \{X_t\}_{t=-\infty}^{\infty} \) with random length \( D_t \) because it includes the case with fixed length.

Due to random length, it is difficult to define weak stationarity of a sequence of time series objects with random length in terms of the mean vector and covariance matrix. Therefore, we first consider an approach based on a transformation of \( \{X_t\}_{t=-\infty}^{\infty} \) into a scalar valued time series. Among possible choices of transformations, data analysts often use the aggregated time series \( \{X_t^a\}_{t=-\infty}^{\infty} \) of \( \{X_t\}_{t=-\infty}^{\infty} \), defined as

\[
X^a_t = \sum_{i=1}^{D_t} X_{t,i}. \tag{1}
\]

For instance, monthly log-returns are defined as sum of daily log-returns each month and those aggregated monthly log-returns are commonly used to calculate the autocorrelation functions and test serial correlation in monthly log-return time series objects.

In order to test whether serial correlation in mean and variance exists in a sequence of time series objects, the aggregated value \( X^a_t \) in Equation (1) can be used. By assuming that \( \{X_{t,1}, \ldots, X_{t,D_t}\}_{t=-\infty}^{\infty} \) for any given \( t \in S_D, -\infty < t < \infty \), is weakly stationary, we can show that the sequence of aggregated values \( \{X^a_t\}_{t=-\infty}^{\infty} \) is also weakly stationary. Then, we can write the hypotheses of testing serial correlation in mean of a sequence of time series objects with random length in terms of the autocorrelation functions \( \rho^a(h) \) of the aggregated time series as follows:

\[
H_0 : \rho^a(1) = \rho^a(2) = \cdots = \rho^a(m) = 0 \text{ vs. } H_1 : \text{Not } H_0,
\]

for a pre-specified positive integer \( m \). The form of \( \rho^a \) can be found in Section 2.2 of supplementary materials. To conduct the test, it is conventional to adopt the Ljung–Box (LB hereafter) test, proposed by Ljung and Box,[8] using the observed aggregated values \( \{X^a_t\}_{t=1}^{T} \). For testing serial correlation in variance of a sequence of time series objects, which is also known as the test for ARCH effect,[14] we employ the LB test based on either the observed squared aggregated values \( \{X^2_t\}_{t=1}^{T} \) or the observed absolute aggregated values \( \{|X^a_t|\}_{t=1}^{T} \). In our simulation study, the LB test based on both the squared and absolute aggregated values are considered.
3. Bootstrap and wild bootstrap-based tests for serial correlation of time series objects

In this section, we propose a nonparametric bootstrap sampling method for testing serial correlation in mean (Section 3.1) and variance (Section 3.2) of a sequence of time series objects. As in Section 2, one could use a representative value from each time series object such as the aggregated value. Instead, we use the within-variation by creating bootstrap time series samples from a sequence of time series objects.

Unfortunately, standard bootstrap tests for serial correlation in mean perform poorly in finite samples and, as a consequence, tend to suffer from severe size distortions in the presence of conditional heteroscedasticity. This is because the standard bootstrap samples fail to mimic the conditional heteroscedasticity of the original data, and thus the standard bootstrap distributions of test statistics under the null hypothesis could be invalid. In Section 3.1, in order to overcome such defects of a standard bootstrap approach, we employ the wild bootstrap-based sampling approach. See Remark 3.1 below for some properties of wild bootstrap methods.

3.1. Wild bootstrap-based tests for serial correlation in mean

For the standard bootstrap test, we consider a bootstrap time series sample \( \{X^*_t\}^\infty_{t=-\infty} \) from a sequence of time series objects \( \{X_t\}^\infty_{t=-\infty} \) by sampling a single value \( X^*_t \) from \( X_t \) independently from the empirical distribution \( P_t \) defined on \( \{X_{t,1}, X_{t,2}, \ldots, X_{t,D_t}\} \).

Under (D) and the assumption that \( \{X_{t,1}, \ldots, X_{t,l}\}^\infty_{t=-\infty} \) for any given \( l_t \in S_D, -\infty < t < \infty \), is weakly stationary with mean \( \mu \) and finite variance \( \sigma^2 \), it is easily seen that \( \{X^*_t\}^\infty_{t=-\infty} \) is also weakly stationary with mean \( \mu \) and variance \( \sigma^2 \).

Then, we can write the hypotheses of testing serial correlation in mean of a sequence of time series objects with random length in terms of the autocorrelation functions \( \rho^*(h) \) of a bootstrap time series sample \( \{X^*_t\}^\infty_{t=-\infty} \) as follows:

\[
H_0 : \rho^*(1) = \rho^*(2) = \cdots = \rho^*(m) = 0 \quad \text{vs.} \quad H_1 : \text{Not } H_0,
\]

for a pre-specified positive integer \( m \). The form of \( \rho^* \) can be found in Section 2.2 of supplementary materials. Below we derive consistency of the \( h \)-lag sample autocorrelation function of a bootstrap time series sample \( \{X^*_t\}^T_{t=1} \), defined as

\[
\hat{\rho}^*(h) = \frac{1}{T-h} \sum_{t=1}^{T-h} (X^*_t - \bar{X}^*_T)(X^*_{t+h} - \bar{X}^*_T).
\]

Theorem 1: Under (D) and the assumption that a sequence of random variables \( \{X_{t,1}, \ldots, X_{t,l}\}^\infty_{t=-\infty} \) for any given \( l_t \in S_D, -\infty < t < \infty \), is strictly stationary and ergodic with mean \( \mu \) and finite variance \( \sigma^2 \), we have, for any fixed positive integer \( h \), \( \hat{\rho}^*(h) \to \rho^*(h) \) a.s. as \( T \to \infty \).

Proof of Theorem 1: Proof is provided in Section 1 of supplementary materials.

Theorem 2: Let

\[
Q^* = T(T + 2) \sum_{h=1}^{m} \frac{\hat{\rho}^*(h)}{T-h}
\]

be the LB test statistic based on the observed bootstrap time series sample \( \{X^*_t\}^T_{t=1} \) for any positive integer \( m \). Under (D) and the assumption that a sequence of random variables \( \{X_{t,1}, \ldots, X_{t,l}\}^\infty_{t=-\infty} \) for any...
Proof of Theorem 2: Proof is provided in Section 1 of supplementary materials.

In what follows, we construct the bootstrap test for serial correlation in mean of a sequence of time series objects with random length. We first obtain a bootstrap time series sample using the observed time series objects \( \{X_t\}_{t=1}^{T} \), and then define the bootstrap sampled autocorrelation function and the bootstrap LB test statistic as follows. For each \( b = 1, \ldots, B \),

**Step 1** sample a single value \( X^*_t \) independently from \( X_t = (X_{t,1}, X_{t,2}, \ldots, X_{t,D_t})' \) based on the empirical distribution \( P_t \) defined on \( \{X_{t,1}, X_{t,2}, \ldots, X_{t,D_t}\} \). Denote the bootstrap time series sample \( \{X^*_t\}_{t=1}^{T} \). Then, calculate the \( b \)th sample autocorrelation function estimate \( \hat{\rho}^{*b}(h) \) for \( h = 1, \ldots, T - 1 \) using \( \{X^*_t\}_{t=1}^{T} \).

**Step 2** For each \( b \), a bootstrap LB test statistic is constructed for any fixed positive integer \( m \),

\[
Q^{*b} = T(T + 2) \sum_{h=1}^{m} \frac{\hat{\rho}^{*b}(h)}{T - h}
\]

based on the bootstrap estimates \( \hat{\rho}^{*b}(h) \) for \( h = 1, \ldots, m \) in Step 1.

We now define an empirical distribution \( F_{TB} \) using the bootstrap LB test statistics \( Q^{*b}, b = 1, \ldots, B \), in Equation (2) to construct a bootstrap test statistic for serial correlation in mean as

\[
F_{TB}(x) = \frac{1}{B} \sum_{b=1}^{B} I(Q^{*b} \leq x),
\]

where \( I \) is the indicator function. In order to derive the asymptotic distribution of \( F_{TB} \), we define an empirical distribution function \( F_B \) of independently and identically distributed chi-square random variables with \( m \) degrees of freedom, denoted by \( \chi^2_b(m) \) for \( b = 1, \ldots, B \) as

\[
F_B(x) = \frac{1}{B} \sum_{b=1}^{B} I(\chi^2_b(m) \leq x).
\]

Then, we propose Kolmogorov–Smirnov-type test with the bootstrap LB test statistics for serial correlation in mean of time series objects in the following theorem.

**Theorem 3:** Let

\[
D_{TB} = \sup_x |F_{TB}(x) - F_{\chi^2(m)}(x)| \quad \text{and} \quad D_B = \sup_x |F_B(x) - F_{\chi^2(m)}(x)|,
\]

where \( F_{\chi^2(m)} \) is a cumulative distribution function of chi-squared random variable with \( m \) degrees of freedom. Under (D) and the assumption that a sequence of random variables \( \{X_{t,1}, \ldots, X_{t,D_t}\}_{t=-\infty}^{\infty} \) for any given \( l_t \in S_D, -\infty < t < \infty \), is independent and identically distributed with \( E[X^4_{t,1}] < \infty \), for a fixed positive integer \( B \), it follows that as \( T \) approaches infinity,

\[
\sqrt{B}D_{TB} \xrightarrow{d} \sqrt{B}D_B
\]
and as $B$ approaches infinity,

$$\sqrt{B}D_B \xrightarrow{d} \sup_t |W^o(t)|,$$

where $W^o$ is a standard Brownian bridge.

**Proof of Theorem 3:** Proof is provided in Section 1 of Supplementary materials.

As demonstrated in our simulation in Section 4, however, the test based on the standard bootstrap sampling might fail to control type I errors particularly for some cases of conditional heteroscedasticity. Hence, the asymptotic distribution of $\chi^2$ in Theorem 3 might not be appropriate due to conditional heteroscedasticity in finite sample. In order to avoid this issue and improve the finite sample performance, we consider the wild bootstrap-based test for serial correlation in mean of time series objects. It can be shown that the wild bootstrap samples do not have serial correlation in mean, but do have the same conditional heteroscedasticity as the original data (cf. Remark 3.1). Therefore, the distribution obtained from the wild bootstrap samples, which takes conditional heteroscedasticity into consideration, can replace $\chi^2$ distribution in Theorem 3.

For the wild bootstrap method, we consider the Rademacher distribution:

$$w_t = \begin{cases} 
1, & \text{with probability } \frac{1}{2}, \\
-1, & \text{with probability } \frac{1}{2}.
\end{cases}$$

(5)

Note that according to Davidson and Flachaire, [15] the Rademacher distribution works well in many applications. The wild bootstrap is now implemented to obtain the wild bootstrap time series samples using the observed time series objects $\{X_t\}_{t=1}^T$ as follows. For each $b = 1, \ldots, B$,

**Step 1** sample a single value $X^{*b}_t$ independently from $X_t$ based on the empirical distribution $P_t$ defined on $\{X_{t,1}, X_{t,2}, \ldots, X_{t,D_t}\}$ and obtain a wild bootstrap sample as $X^*_{t,WB} = X^{*b}_t w_t$, where $w_t$ is a random draw from the Rademacher distribution in Equation (5). Denote the wild bootstrap time series sample $\{X^*_{t,WB}\}_{t=1}^T$.

**Step 2** For each $b$, a wild bootstrap-based LB test statistic is constructed for any fixed positive integer $m$,

$$Q^{*b}_{WB} = T(T+2) \sum_{h=1}^m \frac{\hat{\rho}^{*b}_{WB}(h)}{T-h},$$

(6)

based on the wild bootstrap estimates $\hat{\rho}^{*b}_{WB}(h)$ for $h = 1, \ldots, m$ in Step 1.

**Remark 3.1:** Here, some properties of wild bootstrap methods are provided. Consider any stationary time series, denoted by $\{Y_t\}_{t=-\infty}^\infty$, with autocorrelation functions $\gamma_y(h)$ in mean and $\gamma_{y^2}(h)$ in variance, respectively. Using the Rademacher distribution in Equation (5), we can obtain the wild bootstrap sampled time series $\{Y^*_t := Y_t w_t\}_{t=-\infty}^\infty$; then, the autocorrelation functions of $\{Y^*_t\}_{t=-\infty}^\infty$ in mean and variance for any positive integer $h$ are given as follows:

$$\text{Cov}(Y^*_t, Y^*_t) = 0,$$

$$\text{Cov}(Y^*_{t+h}, Y^*_{t+h}) = \text{Cov}(Y^2_t, Y^2_{t+h}) = \gamma_{y^2}(h),$$

since $\{w_t\}$ is assumed to be a sequence of IID random variables and independent of $\{Y_t\}$.

From the equation above, irrespective of the original time series data, the wild bootstrap sampled time series is a white noise process in mean, but has the same autocorrelation function in variance of the original time series data. These properties allow the wild bootstrap sampled time series to mimic
the original time series with the same conditional heteroscedasticity. Therefore, the wild bootstrap enables us to have the exact finite sample distribution of the test statistics under the null distribution with the conditional heteroscedasticity, and thus we can avoid the influence of the conditional heteroscedasticity on the test for serial correlation in mean of time series data.

Using the wild bootstrap-based LB test statistic, we define

$$F_{TWB}(x) = \frac{1}{B} \sum_{b=1}^{B} I_{(Q_{WB}^b \leq x)}$$

and

$$F_{WB}(x) = \frac{1}{B} \sum_{b=1}^{B} I_{(\chi^2_b(m) \leq x)},$$

where the empirical distribution function $F_{WB}$ are defined by using another independent and identically distributed chi-square random variable with $m$ degrees of freedom, denoted by $\chi^2_b(m)$ for $b = 1, \ldots, B$, which is different from the one in Theorem 3. Then, we have the following theorem. The proof of the theorem is omitted since it can be similarly derived as Theorem 3, Smirnov [7] and the references therein.

**Theorem 4:** Let

$$D_{TWB} = \sup_x \left| F_T(x) - F_{TWB}(x) \right| \quad \text{and} \quad D_{WB} = \sup_x \left| F_B(x) - F_{WB}(x) \right|.$$ 

Under (D) and the assumption that a sequence of random variables $\{X_t, \ldots, X_{t+l_t}\}_{t=-\infty}^{\infty}$ for any given $l_t \in S_D$, $-\infty < t < \infty$, is independent and identically distributed with $E[X_{1,1}^4] < \infty$, for a fixed positive integer $B$, it follows that as $T$ approaches infinity,

$$\sqrt{\frac{B}{2}} D_{TWB} \xrightarrow{d} \sqrt{\frac{B}{2}} D_{WB}$$

and as $B$ approaches infinity,

$$\sqrt{\frac{B}{2}} D_{WB} \xrightarrow{d} \sup_t |W^\alpha(t)|,$$

where $W^\alpha$ is a standard Brownian bridge.

By Theorems 3 and 4, the null hypothesis for serial correlation in mean of time series objects is rejected at the significance level $\alpha$ for sufficiently large $T$ and $B$ if $\sqrt{BD_T} > C_\alpha$, or $\sqrt{B/2} D_{TWB} > C_\alpha$, where $C_\alpha$ is a $100(1 - \alpha)$ percentile of $\sup_t |W^\alpha(t)|$. We will compare the finite sample performance of $\sqrt{BD_T}$ and $\sqrt{B/2} D_{TWB}$ in Section 4.

### 3.2. Bootstrap-based test for serial correlation in variance

In this subsection, we consider the test for serial correlation in variance of a sequence of time series objects with random length. Here, we briefly describe our proposed bootstrap method, since it is similar to the test for serial correlation in mean of a sequence of time series objects. Suppose again that we have a bootstrap time series sample $\{X_t^b\}_{t=1}^T$ as in Section 3.1. For testing serial correlation in variance, we first obtain a sample autocorrelation function of a power transformed bootstrap time series sample $\{[X_t^b]^\nu\}_{t=1}^T$. A typical value for $\nu$ is 1 or 2, which results in the sample autocorrelation function of absolute and squared bootstrap time series samples, respectively.

Let us assume that $\{[X_t^b]^\nu\}_{t=-\infty}^{\infty}$ is weakly stationary. Then, we express the hypotheses of testing serial correlation in variance of a sequence of time series objects in terms of the $h$-lag autocorrelation function $\rho^*_\nu(h)$ of $\{[X_t^b]^\nu\}_{t=1}^T$ as

$$H_0 : \rho^*_\nu(1) = \rho^*_\nu(2) = \cdots = \rho^*_\nu(m) = 0 \quad \text{vs.} \quad H_1 : \text{Not } H_0,$$

for a pre-specified positive integer $m$. Similarly as in Theorems 1–3, we have the following asymptotic results. The proofs are omitted since they are similar to those of Theorems 1–3.
Theorem 5: Under (D) and the assumption that a sequence of random variables \(\{|X_{t,1}|^\nu, \ldots, |X_{t,l}|^\nu\}_{t=-\infty}^{\infty}\) for any given \(l \in S_D, -\infty < t < \infty\), is strictly stationary and ergodic with finite variance, we have, for any fixed positive integer \(h\), \(\hat{\rho}_v^*(h) \rightarrow \rho_v^*(h)\) a.s. as \(T \rightarrow \infty\).

Theorem 6: Let

\[ Q_v^* = T(T + 2) \sum_{h=1}^{m} \frac{\hat{\rho}_v^*(h)}{T - h} \]

be the LB test statistic based on the power transformed bootstrap time series sample \(\{|X_{t,1}|^\nu\}_{t=1}^{T}\). Under (D) and the assumption that a sequence of random variables \(\{X_{t,1}, \ldots, X_{t,l}\}_{t=-\infty}^{\infty}\) for any given \(l \in S_D, -\infty < t < \infty\), is independent and identically distributed with \(E[|X_{1,1}|^{4\nu}] < \infty\), it follows that as \(T \rightarrow \infty\), \(Q_v^*\) converges in distribution to a chi-square random variable with \(m\) degrees of freedom.

Below we define, for each \(b\), a bootstrap LB test statistic based on the power transformed bootstrap time series sample \(\{|X_{t,1}|^{b\nu}\}_{t=1}^{T}\) as

\[ Q_v^{*b} = T(T + 2) \sum_{h=1}^{m} \frac{\hat{\rho}_v^{*b}(h)}{T - h} \]

using the bootstrap estimates \(\hat{\rho}_v^{*b}(h)\) for \(h = 1, \ldots, m\) and \(b = 1, \ldots, B\). Similarly, we define an empirical distribution \(F_{v,TB}\) with \(B\) bootstrap LB test statistics \(Q_v^{*b}\) for \(b = 1, \ldots, B\) as

\[ F_{v,TB}(x) = \frac{1}{B} \sum_{b=1}^{B} I(Q_v^{*b} \leq x). \]

Then, we have the following two theorems.

Theorem 7: Let

\[ D_{v,TB} = \sup_x |F_{v,TB}(x) - F_{X^2(m)}(x)| \quad \text{and} \quad D_B = \sup_x |F_B(x) - F_{X^2(m)}(x)|, \]

where \(F_{X^2(m)}\) is a distribution function of chi-squared random variable with \(m\) degrees of freedom. Under (D) and the assumption that a sequence of random variables \(\{X_{t,1}, \ldots, X_{t,l}\}_{t=-\infty}^{\infty}\) for any given \(l \in S_D, -\infty < t < \infty\), is independent and identically distributed with \(E[|X_{1,1}|^{4\nu}] < \infty\), for a fixed positive integer \(B\), it follows that as \(T\) approaches infinity,

\[ \sqrt{BD_{v,TB}} \xrightarrow{d} \sqrt{BD_B} \]

and as \(B\) approaches infinity,

\[ \sqrt{BD_B} \xrightarrow{d} \sup_t |W^o(t)|, \]

where \(W^o\) is a standard Brownian bridge.

Theorem 7 enables us to reject the null hypothesis of no serial correlation in variance of a sequence of time series objects with random length at the significance level \(\alpha\) for sufficiently large \(T\) and \(B\) if \(\sqrt{BD_{v,TB}} > C_\alpha\), where \(C_\alpha\) is a \(1 - \alpha\) percentile of \(\sup_t |W^o(t)|\).
4. Simulation study

In this section, we present the finite sample performance of our proposed bootstrap testing methods for time series objects through Monte Carlo simulation. To measure the empirical sizes of the tests for serial correlation in mean and variance, we generate IID random samples from the following distributions to mimic 5-minute observations in daily time series objects and daily observations in monthly time series objects:

(a) $N(0,1)$
(b) $T(df)$ – $t$ distribution with $df$ degrees of freedom
(c) $ST(df)$ – skewed $t$ distribution with $df$ degrees of freedom and shape parameter $-4$.

Here, we consider the skewed and heavy-tailed distributions of (b)–(c) to see how the proposed bootstrap tests are influenced by non-normality. In order to satisfy the moment conditions in Theorems 3–7, we set $df = 5$ and $df = 10$ in (b)–(c) for testing serial correlation in mean and variance, respectively. For the serial correlation tests in mean, we also consider GARCH(1,1) models (d) below to see the empirical sizes in the presence of conditional heteroscedasticity:

(d) GARCH(1,1) models with the parameter $(\omega, \alpha, \beta)$:

$$X_t = \sigma_t \epsilon_t, \quad \sigma_t^2 = \omega + \alpha X_{t-1}^2 + \beta \sigma_{t-1}^2.$$

To measure the empirical powers, for serial correlation in mean of time series objects, we generate AR(1) and AR(1)-GARCH(1,1) models:

(e) AR(1) models with the parameter $\phi$:

$$X_t = \phi X_{t-1} + \epsilon_t,$$

(f) AR(1)-GARCH(1,1) models with the parameter $(\phi, \omega, \alpha, \beta)$:

$$X_t = \phi X_{t-1} + \epsilon_t, \quad \epsilon_t = \sigma_t \eta_t, \quad \sigma_t^2 = \omega + \alpha \epsilon_{t-1}^2 + \beta \sigma_{t-1}^2$$

and for serial correlation in variance of time series objects, we generate GARCH(1,1) models in (d) with different parameter values. We consider $\phi = 0.95$ and $(\omega, \alpha, \beta) = (0.10, 0.10, 0.89)$ for the parameter values of AR(1) and GARCH(1,1) models, respectively. For samples of monthly time series objects, we select a random length $D_t$ from $SD = \{20, 21, 22\}$ with equal probability $\frac{1}{3}$.

In all cases, we generate two different samples of time series objects with 70 intra-day observations each day and 20–22 business day observations each month. For the duration of a time period, we consider 1, 2, 4 and 8 years, which correspond to 250, 500, 1000, 2000 days for the samples of daily time series objects, and 12, 24, 48 and 96 months for the samples of monthly time series objects, respectively. We set $B = 100, 200, 500, 1000$ and $B = 50, 100$ for testing serial correlation in mean and variance, respectively. We repeat this process 1000 times for each simulation setting. To save space we present the results with 500 and 2000 days, 24 and 96 months, and $B = 500, 1000$ for testing in mean and $B = 50, 100$ for variance in this section. The complete results are presented in Section 3 of Supplementary materials.

In order to assess the testing performance for serial correlation in mean of time series objects, we compare our proposed Kolmogorov–Smirnov test based on the bootstrap LB test statistic (KS) in Theorem 3 and the wild bootstrap LB test statistic (WB) in Theorem 4 with the conventional LB test statistic (LB) of aggregated time series data in Section 2. For serial correlation in variance (ARCH effect hereafter) of time series objects, we compare our proposed Kolmogorov–Smirnov test based on the bootstrap LB test statistic of squared (KSS) and absolute (KSA) samples in Theorem 7 with the LB test of squared (LBS) and absolute (LBA) aggregated time series data in Section 2.
Empirical sizes of tests for serial correlation in mean are presented in Figures 2–3. There is no size distortion for WB test except the cases of the skewed and heavy-tailed distributions. On the other hand, it is evident that LB and KS tests suffer from the severe size distortion in the presence of conditional heteroscedasticity. The empirical powers for serial correlation in mean of AR(1) and AR(1)-GARCH(1,1) time series objects are reported in Figures 4–5. We only consider WB due to the severe size distortions of KS. It is observed that the proposed WB method provides excellent power and the power approaches 1 as the sample size increases. Another interesting finding is that power increases when $B$ increases in all cases.
Figure 4. Powers of LB, KS and WB tests for serial correlation in mean of daily time series objects with 70 observations each day are plotted. The parentheses of the KS and WB methods indicate the bootstrap repetition $B$. The dashed line in each plot indicates the significance level .05.

Figure 5. Powers of LB, KS and WB tests for serial correlation in mean of monthly time series objects with any randomly chosen one of 20–22 observations each month are plotted. The parentheses of the KS and WB methods indicate the bootstrap repetition $B$. The dashed line in each plot indicates the significance level .05.
Figures 6–9 display the empirical sizes and powers of tests for ARCH effect of time series objects. Empirical sizes in Figures 6–7 show that KSS suffers from severe size distortions, although size distortions slowly disappear as the sample size increases. On the other hand, the KSA method achieves excellent empirical sizes compared to LBS and LBA in all cases. The empirical powers of tests for ARCH effect of GARCH(1,1) models are reported in Figures 8–9. Here, we only consider KSA due to the severe size distortions of KSS. Most of the results are consistent with the previous cases for testing serial correlation in mean of AR(1) and AR(1)-GARCH(1,1) time series objects. We can see that our KSA method provides excellent empirical powers in all cases compared to conventional LBA and
LBS. Similar to the WB method for testing serial correlation in mean, the power quickly approaches 1 as sample size increases, and $B$ gets larger.

The summary of findings from our simulation study is consistent with the theoretical development of our test statistics. First, the simulation results strongly support the superior performance of our WB and KSA test statistics versus the conventional LB test statistic in most cases. Second, the persistence of serial correlation in mean and variance leads to good power of our WB and KSA test statistics. Finally, the power in all cases increases when $B$ increases. Therefore, in order to achieve a more powerful test and to avoid size distortion, it is highly recommended to increase $B$ as the sample size increases.

5. Real data analysis

For an example of applying our methods to real data, we consider the daily and monthly Korean Composite Stock Price Index 200 (KOSPI200) time series objects from 2 January 2004 to 30 December 2011. The daily and monthly KOSPI200 time series objects consist of around 70 observations each day (at approximately 5-minute intervals) and around 20 daily observations each month, respectively. Gaussian quasi-maximum likelihood estimates of GARCH models, given in Table 1, show that the log-returns of 5-minute intra day and daily KOSPI200 time series are highly persistent.

The proposed tests are applied to test serial correlation in mean and variance of KOSPI200 time series objects. The results of LB, KS and WB tests for serial correlation in mean of daily and monthly KOSPI200 time series objects are given in Table 2. It is seen that both LB and WB tests do not reject the null hypothesis, unlike KS test. Given high persistence of 5-minute intra day and daily KOSPI200
Table 1. Estimates of GARCH(1,1) models for five minute and daily KOSPI200 observations.

<table>
<thead>
<tr>
<th></th>
<th>(\hat{\omega})</th>
<th>(\hat{\alpha})</th>
<th>(\hat{\beta})</th>
<th>(\hat{\alpha} + \hat{\beta})</th>
</tr>
</thead>
<tbody>
<tr>
<td>Five minute</td>
<td>0.0001</td>
<td>0.0081</td>
<td>0.9881</td>
<td>0.9962</td>
</tr>
<tr>
<td>Daily</td>
<td>0.0001</td>
<td>0.0389</td>
<td>0.9525</td>
<td>0.9914</td>
</tr>
</tbody>
</table>

Table 2. LB, KS and WB tests for serial correlation in mean of daily and monthly KOSPI200 time series objects.

<table>
<thead>
<tr>
<th></th>
<th>KS</th>
<th>WB</th>
</tr>
</thead>
<tbody>
<tr>
<td>B</td>
<td>100</td>
<td>200</td>
</tr>
<tr>
<td>Daily</td>
<td></td>
<td></td>
</tr>
<tr>
<td>Teststatistic</td>
<td>7.486</td>
<td>5.780</td>
</tr>
<tr>
<td>P-value</td>
<td>0.380</td>
<td>0.000</td>
</tr>
<tr>
<td>Monthly</td>
<td></td>
<td></td>
</tr>
<tr>
<td>Teststatistic</td>
<td>8.531</td>
<td>2.640</td>
</tr>
<tr>
<td>P-value</td>
<td>0.074</td>
<td>0.000</td>
</tr>
</tbody>
</table>

Table 3. LBS, LBA and KSA tests for serial correlation in variance of daily and monthly KOSPI200 time series objects.

<table>
<thead>
<tr>
<th></th>
<th>KSA</th>
</tr>
</thead>
<tbody>
<tr>
<td>B</td>
<td>LBS</td>
</tr>
<tr>
<td>Daily</td>
<td></td>
</tr>
<tr>
<td>Teststatistic</td>
<td>443.7</td>
</tr>
<tr>
<td>P-value</td>
<td>0.000</td>
</tr>
<tr>
<td>Monthly</td>
<td></td>
</tr>
<tr>
<td>Teststatistic</td>
<td>2.670</td>
</tr>
<tr>
<td>P-value</td>
<td>0.614</td>
</tr>
</tbody>
</table>

time series in Table 1 and the evidence of severe size distortion of LB and KS tests in the presence of strong conditional heteroscedasticity in Figures 2 and 3, we conclude that WB test is more reliable in this case, and thus the null hypothesis of no serial correlation in mean is favourable.

Table 3 summarizes the results of LBS, LBA and KSA tests for serial correlation in variance of the daily and monthly KOSPI200 time series objects. At the 5% significance level, all the tests reject the null hypothesis of no ARCH effect, and we conclude that the serial correlation in variance exists in the daily KOSPI200 time series objects. On the other hand, only our KSA test rejects the null hypothesis of no ARCH effect in the monthly KOSPI200 time series objects. Since our KSA is a more powerful test than the others (as evident in Figure 9), we conclude that the alternative hypothesis of ARCH effect appears more favourable.

6. Supplementary materials

Supplementary materials: Details of Sections 2, proofs of Theorems 1–3, and complete results of Section 4 are provided. (supplement.pdf)

R-code: R-codes to perform our tests procedures in the simulation study are provided. (Figure1.R, Figure2.R, Figure3.R, Figure4.R, Figure5.R, Figure6.R, Figure7 and R, Figure8.R)

Readme: R-codes are fully described. (readme.txt)

Disclosure statement

No potential conflict of interest was reported by the authors.
**Funding**

The first author’s research was supported by Basic Science Research Program through the National Research Foundation of Korea (NRF) funded by the Ministry of Education [NRF-2016R1D1A1B01009508].

**References**


